PROOF OF THE ATIYAH-SINGER INDEX THEOREM USING THE RESCALING OF THE SPIN-DIRAC OPERATOR AND ITS ASSOCIATED HEAT KERNEL

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ABSTRACT. This is a study note on the heat kernel proof of the Atiyah-Singer index theorem à la Getzler [2]. Main references consulted were Roe [5] and Freed [3].

1 PRELIMINARIES

1.1 SET-UP. Let M be a compact oriented manifold M with dimension n. Let g be a riemannian metric on M. Let $\mathrm{Fr}_{\mathrm{SO}}(TM)$ be the principal $\mathrm{SO}(n)$-bundle of oriented frames of the tangent bundle TM of M. We assume that M admits a spin structure, that is, there is a principal $\mathrm{Spin}(n)$-bundle $\mathrm{P}_{\mathrm{Spin}}(M)$ over M and a bundle map

$$\rho : \mathrm{P}_{\mathrm{Spin}}(M) \to \mathrm{Fr}_{\mathrm{SO}}(TM)$$

such that

$$\rho(s \cdot p) = \pi(s) \cdot \rho(p),$$

where $p \in \mathrm{P}_{\mathrm{Spin}}(M)$, $s \in \mathrm{Spin}(n)$, and $\pi : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the spin double cover.

Let $\mathbb{C}\mathrm{l}(TM)$ denote the (complex) Clifford bundle over M; its fiber over $x \in M$ is the Clifford algebra $\mathbb{C}\mathrm{l}(T_xM)$ constructed from the tangent space T_xM and the metric. Using the spin structure and the Borel mixing construction, we can always construct a vector bundle $E \to M$ whose fiber over $x \in M$ is a Clifford module over $\mathbb{C}\mathrm{l}(T_xM)$. Assume that the bundle E is also equipped with a hermitian metric $(\ ,\ )$. We can find a connection ∇ on E such that, for any vector fields X, Y on M and sections σ_1, σ_2 of E,

(i) the Clifford action $c : \mathbb{C}\mathrm{l}(TM) \to \mathrm{End}(E)$ is skew-adjoint,

$$(c(X)\sigma_1, \sigma_2) = -(\sigma_1, c(X)\sigma_2),$$

(ii) the connection ∇ is compatible with the hermitian metric,

$$X(\sigma_1, \sigma_2) = (\nabla_X \sigma_1, \sigma_2) + (\sigma_1, \nabla_X \sigma_2),$$

(iii) the connection ∇ is compatible with the riemannian connection (also denoted by ∇) on M,

$$[\nabla_X, c(Y)] = c(\nabla_X Y).$$

Then the **geometric** (or the **riemannian**) **Dirac operator** is defined as the following composition:

$$D : \Gamma(E) \xrightarrow{\nabla} \Gamma(T^*M \otimes E) \xrightarrow{g} \Gamma(TM \otimes E) \longrightarrow \Gamma(E).$$

Here, $\Gamma(E)$ denotes the space of sections of E, and the last map is provided by the Clifford action. In terms of a local orthonormal frame $e_1, \ldots, e_n$ for the tangent bundle TM, we have

$$D = \sum_{i=1}^{n} c(e_i)\nabla_{e_i}.$$

1.2 ANALYTIC PROPERTIES OF D. We summarize the analytic properties of the Dirac operator D as follows (See [5, Ch. 5, 7].): It is an elliptic, (essentially) self-adjoint, Fredholm operator on the space $L^2(E)$ of square integrable sections of the Clifford module bundle E. The eigenvectors of D form a complete orthonormal basis for $L^2(E)$. Each eigenvalue comes with finite multiplicity. The heat diffusion operator e^{-tD^2} is admits an integral kernel k_t so that

$$(e^{-tD^2}s)(x) = \int_M k_t(x, y)s(y)\,\mathrm{vol}_y, \tag{1.3}$$

where vol_y is the riemannian volume form at y. Let p_1, p_2 be the projections of $M \times M$ onto the first and the second component, respectively. Let $E \boxtimes E^* := p_1^*E \otimes p_2^*E^*$. Then $t \mapsto k_t$ is a smooth map from $]0, \infty[$ to the space of sections of $E \boxtimes E^*$. The kernel k_t is in fact the fundamental solution of the heat equation associated to D. That means

$$(\partial_t + D_x^2)k_t(x, y) = 0, \tag{1.4}$$

where the subscript in D_x denotes differentiation with respect to the x-variable, and it behaves like the delta distribution under the limit $t \longrightarrow 0+$ in the sense that, for any smooth section s of E,

$$\lim_{t\to 0+} \int_M k_t(x, y)s(y)\,\mathrm{vol}_y = s(x)$$

under the uniform topology.

Suppose E is $\mathbb{Z}/2\mathbb{Z}$-graded so that $\Gamma(E) = \Gamma(E)^+ \oplus \Gamma(E)^-$. Let ε be the grading operator for $\Gamma(E)$ so that $\Gamma(E)^\pm$ are the ± 1-eigenspaces of ε. We assume that D anti-commutes with ε, which is to say that D is an odd operator. Then D decomposes as

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}.$$

We defined the (**graded**) **index** of D as

$$\operatorname{ind}_s D := \dim\ker(D_+) - \dim\ker(D_-).$$

The **super-trace** of an operator on $\Gamma(E)$ is the usual trace precomposed with the grading operator. Owing to the McKean-Singer formula [4], the index of D can be obtained from the super-trace of e^{-tD^2}:

$$\operatorname{ind}_s D = \operatorname{tr}_s e^{-tD^2} = \operatorname{tr}(\varepsilon e^{-tD^2}). \tag{1.5}$$

In terms of the heat kernel, the above can be written as

$$\operatorname{ind}_s D = \int_M \operatorname{tr}_s(k_t(y,y))\operatorname{vol}_y. \tag{1.6}$$

Note that the left-hand side is independent of t. Thus, the above equation should hold even in the limit of $t \to 0+$. Under that limit the heat kernel has an asymptotic expansion

$$k_t(x,y) \sim \frac{1}{(4\pi t)^{n/2}} \sum_{j=0}^{\infty} a(x,y)_j t^j \tag{1.7}$$

where $n = \dim M$. This leads to

$$\operatorname{ind}_s D = \lim_{t\to 0+} \int_M \operatorname{tr}_s k_t(y,y)\operatorname{vol}_y \tag{1.8}$$

$$= \lim_{t\to 0+} \frac{1}{(4\pi t)^{n/2}} \sum_{j=0}^{\infty} \Big(\int_M \operatorname{tr}_s a(y,y)_j \, dy\Big) t^{j-n/2}. \tag{1.9}$$

From the finiteness of the left-hand side, we conclude that the index is zero if n is odd. So, in the rest of the presentation, we shall assume that the dimension of M is even. Then, we must have

$$0 = \int_M \operatorname{tr}_s a(y,y)_j \operatorname{vol}_y \tag{1.10}$$

for $0 \leqslant j \leqslant \frac{n}{2} - 1$, and

$$\operatorname{ind}_s D = \frac{1}{(4\pi)^{n/2}} \int_M \operatorname{tr}_s a(y,y)_{n/2} \operatorname{vol}_y. \tag{1.11}$$

1.12 HEAT KERNEL IN NORMAL COORDINATES. To evaluate the integral 1.11, we need to know the asymptotic expansion of the heat kernel along the diagonal, $k_t(y,y)$. Let us fix $y \in M$. Take the normal coordinates (exponential chart) about y. The normal coordinates use the exponential map $\operatorname{Exp}_y : T_yM \to M$ to describe points near y. We wish to write down an expression for

$$k_t(X) := k_t(\operatorname{Exp}_y X, y) \in \operatorname{Hom}(E_y, E_{\operatorname{Exp}_y X}).$$

In this notation we have suppressed the dependence on y. Using parallel transport along the geodesic connecting y and $\operatorname{Exp} X$ we can identify $E_{\operatorname{Exp}_y X}$ with E_y. This gives means to identify the value of the heat kernel $k_t(X)$ with an element of $\operatorname{End}(E_y)$. Since we assume that M is of even dimension, the spinor representation $\mathbb{S}$ for $\mathbb{Cl}(T_y M)$ is naturally $\mathbb{Z}/2\mathbb{Z}$-graded, and any spinor module is isomorphic to $\mathbb{S} \otimes V$ where V is some auxiliary vector space on which the Clifford algebra acts trivially. Hence, we may assume that

$$E = S \otimes F$$

where S is the spinor bundle and F is some twisting bundle. And we may take the value of $k_t(X)$ to be in $\mathbb{Cl}(T_y X) \otimes \operatorname{End}(F_y)$.

Let us write $k_t(X)$ using a basis for $\mathbb{Cl}(T_y X)$. Let $\{e_1, \ldots, e_n\}$ be the selected orthonormal basis for $T_y M$. For each subset $I \subseteq \{1, \ldots, n\}$, define $e_I = 1$ if $I = \varnothing$, and $e_I = e_{i_1} \cdots e_{i_k}$ if $I = \{i_1, \ldots, i_k\}$ with $i_1 < \cdots < i_k$. Then the asymptotic expansion 1.7 can be written in the following form:

$$k_t(X) \sim \frac{1}{(4\pi t)^{n/2}} \sum_{j=0}^{\infty} \sum_{I} a(X)_{j,I} e_I t^j. \tag{1.13}$$

The coefficients $a(X)_{j,I}$ are $\operatorname{End}(F_y)$-valued. Our ultimate goal is to evaluate the integral 1.11; so we are interested in the super-trace of $k_t(X)$ at $X = 0$, or rather, its asymptotic expansion

$$\operatorname{tr}_s k_t(0) \sim \frac{1}{(4\pi t)^{n/2}} \sum_{j=0}^{\infty} \sum_{I} \operatorname{tr}_F(a(0)_{j,I}) \operatorname{tr}_{\mathbb{S}}(e_I) t^j. \tag{1.14}$$

Here, tr_F is the ordinary trace for $\operatorname{End}(F_y)$, and $\operatorname{tr}_{\mathbb{S}}$ is the super-trace for $\operatorname{End}(\mathbb{S})$. Now, the super-trace $\operatorname{tr}_{\mathbb{S}} e_I$ is nonvanishing only for e_I of top filtration degree because, if $I \neq \{1, \ldots, n\}$, then e_I is a super-commutator: $e_I = -\frac{1}{2}[e_I e_i, e_i]_s$ for any $i \notin I$. But, if $I = \{1, \ldots, n\}$ then, using the fact that the grading operator for $\mathbb{Cl}(n)$ is provided by the element $i^{n/2} e_1 \cdots e_n$, we have $\operatorname{tr}_{\mathbb{S}}(e_1 \cdots e_n) = \operatorname{tr}(i^{n/2} e_1 \cdots e_n e_1 \cdots e_n) = i^{n/2}(-1)^{n(n+1)/2} \dim(\mathbb{S}) = i^{n/2}(-1)^{n/2} 2^{n/2}$. Thus,

$$\operatorname{tr}_{\mathbb{S}}(e_I) = \begin{cases} (-2i)^{n/2}, & \text{if } I = \{1, 2, \ldots, n\}; \\ 0, & \text{otherwise.} \end{cases} \tag{1.15}$$

So

$$\operatorname{tr}_s k_t(0) \sim \frac{(-2i)^{n/2}}{(4\pi t)^{n/2}} \sum_{j=0}^{\infty} \operatorname{tr}_F a(0)_{j,\{1,\ldots,n\}} t^j.$$

So a refined version of Eq. (1.10) is

$$0 = \int_M \operatorname{tr}_F a(y, y)_{j,\{1,\ldots,n\}} \operatorname{vol}_y \tag{1.16}$$

for $0 \leqslant j \leqslant \frac{n}{2} - 1$. And Eq. (1.11) now takes the form

$$\operatorname{ind}_s D = \frac{(-2i)^{n/2}}{(4\pi)^{n/2}} \int_M \operatorname{tr}_F a(y,y)_{\frac{n}{2},\{1,\dots,n\}} \operatorname{vol}_y. \tag{1.17}$$

To evaluate the quantity $\operatorname{tr}_F a(y,y)_{\frac{n}{2},\{1,\dots,n\}}$, we need to investigate further the behavior of the heat kernel $k_t(X)$ in the limit of $t \longrightarrow 0+$.

1.18 GETZLER RESCALING. The operator e^{-tD^2} is related to the Boltzmann factor with temperature $1/t$. The limit $t \longrightarrow 0+$ is the high temperature limit. Employing the language of physics, a physical system under the high temperature limit behaves more like a classical system, and the interactions among its constituents become localized. To mimic this limit we shall introduce two rescaling maps. Let λ denote a nonnegative real number, serving as the rescaling parameter.

The first rescaling we define is for the metric g on M,

$$g_\lambda := \lambda^2 g.$$

Denote by $\mathbb{C}\mathrm{l}(T_yM)_\lambda$ the Clifford algebra generated by T_yM with respect to to the rescaled inner product g_λ. Hence, when $\lambda = 1$, we have the usual $\mathbb{C}\mathrm{l}(T_yM)$. When $\lambda = 0$, we simply have the exterior algebra:

$$\mathbb{C}\mathrm{l}(T_yM)_0 = \wedge T_yM. \tag{1.19}$$

For $\lambda > 0$, there is an algebra isomorphism

$$\begin{aligned} U_\lambda : \mathbb{C}\mathrm{l}(T_yM)_1 &\longrightarrow \mathbb{C}\mathrm{l}(T_yM)_\lambda \\ e_I &\longmapsto \lambda^{-|I|} e_I. \end{aligned} \tag{1.20}$$

Then,

$$\lim_{\lambda \to 0+} \lambda^{|I|} U_\lambda(e_I) = \hat{e}_I \in \wedge T_yM, \tag{1.21}$$

where $\hat{e}_I$ is defined exactly as e_I except using the exterior multiplication.

The second rescaling we define is the map

$$\begin{aligned} T_\lambda : T_yM &\longrightarrow T_yM \\ X &\longmapsto \lambda X. \end{aligned} \tag{1.22}$$

The pullback $T_\lambda^* : C^\infty(T_yM) \longrightarrow C^\infty(T_yM)$ will serve as an instrument for localization.

Let us apply U_λ and the pullback T_λ^* on $k_t(X)$. Then the asymptotic expansion 1.13 gives us

$$U_\lambda T_\lambda^* k_t(X) \sim \frac{1}{(4\pi t)^{n/2}} \sum_{j,I} \lambda^{-|I|} a(\lambda X)_{j,I} e_I t^j. \tag{1.23}$$

Remember that we are ultimately interested in the coefficients $a(X)_{j,I}$ with $|I| = n$ and taking the limit $\lambda \longrightarrow 0$. But then the factor $\lambda^{-|I|}$ in front of $a(X)_{j,|I|=n}$ would blow up. So consider for the moment the function $k_t^\lambda(X)$ whose asymptotic expansion is

$$k_t^\lambda(X) \sim \frac{1}{(4\pi t)^{n/2}} \sum_{j,I} \lambda^{2j-|I|} a(\lambda X)_{j,I} e_I t^j. \tag{1.24}$$

In fact, the above expansion can be obtained from the expansion 1.23 by first making the substitution

$$t \mapsto \lambda^2 t$$

and then multiplying by λ^n. This motivates us to consider the function

$$k_t^\lambda := \lambda^n U_\lambda T_\lambda^* k_{\lambda^2 t}. \tag{1.25}$$

We shall see in Cor. 2.14 that this is the heat kernel of a rescaled Dirac operator.

2 PROOF OF THE INDEX THEOREM

2.1 MAIN IDEA.

Our aim is to show the following:

In the limit of $\lambda \longrightarrow 0+$, the super-trace of the rescaled heat kernel k_t^λ defined by Eq. (1.25) will lead us to the integrand in Eq. (1.17) for $\operatorname{ind}_s D$. More precisely, we will prove the following:

(A) The rescaled function k_t^λ is the heat kernel of some rescaled Dirac operator D_λ.

(B) The limit $D_0^2 := \lim_{\lambda \to 0+} D_\lambda^2$ exists (under the strong operator topology).

(C) The asymptotic expansion for the heat kernel k_t^0 of D_0^2 can be obtained by taking the limit $\lambda \longrightarrow 0+$ of the asymptotic expansion for k_t^λ. The asymptotic expansion of the super-trace of k_t^0 thus obtained is

$$\operatorname{tr}_s k_t^0 \sim (2\pi i)^{-n/2} \operatorname{tr}_F a(0)_{n/2,(0,1,\dots,n)}. \tag{2.2}$$

(D) The kernel k_t^0 can be explicitly calculated:

$$k_t^0(X) = \frac{1}{(4\pi t)^{n/2}} \det{}^{\frac{1}{2}}\left(\frac{t\Omega^M/2}{\sinh t\Omega^M/2}\right) e^{-\frac{1}{4t} g\left(\frac{t\Omega^M}{2} \coth \frac{t\Omega^M}{2} X, X\right)} e^{-t\Omega^F}, \tag{2.3}$$

where Ω^M is the section of $\operatorname{End}(TM) \otimes \wedge^2 TM$ corresponding to the curvature 2-form of the tangent bundle under the identification of $\wedge^2 T^*M$ with $\wedge^2 TM$ by the metric; Ω^F is defined similarly for the auxiliary bundle $F \longrightarrow M$.

(E) Calculating the left-hand side of the asymptotic equality 2.2 leads to

$$\operatorname{tr}_F a(0)_{n/2,(0,1,\dots,n)} \mathrm{vol} = (2\pi i)^{n/2} \hat{A}(M) \operatorname{ch}(F)\Big|_{n\text{-form}}. \tag{2.4}$$

Combining Eq. (2.4) and Eq. (1.17) yields the Atiyah-Singer index theorem,

$$\operatorname{ind}_s D = \int_M \hat{A}(M)\,\mathrm{ch}(F)\Big|_{n\text{-form}}.$$

2.5 PROOF OF (A). The rescaled heat kernel k_t^λ is related to the original heat kernel k_t by

$$k_t^\lambda = R_\lambda k_{\lambda^2 t},$$

where $R_\lambda := \lambda^n U_\lambda T_\lambda^*$. The kernel $k_{\lambda^2 t}$ satisfies the differential equation

$$\Big(\frac{1}{\lambda^2}\partial_t + D^2\Big)k_{\lambda^2 t} = 0.$$

So the rescaled heat kernel k_t^λ satisfies

$$R_\lambda\Big(\frac{1}{\lambda^2}\partial_t + D^2\Big)R_\lambda^{-1}k_t^\lambda = 0.$$

Or equivalently,

$$(\partial_t + \lambda^2 R_\lambda D^2 R_\lambda^{-1})k_t^\lambda = 0.$$

It follows that k_t^λ is the heat kernel for the rescaled Dirac operator

$$D_\lambda^2 := \lambda^2 R_\lambda D^2 R_\lambda^{-1}. \tag{2.6}$$

□

2.7 PROOF OF (B). To calculate $\lim_{\lambda\to 0+} D_\lambda^2$, we adopt once again the normal coordinates and the synchronous frame for the bundle $E = S \otimes F$. Write

$$\nabla_{\partial_i} = \partial_i + \omega_i + A_i$$

where ω_i, A_i are the Christoffel symbols for $\mathbb{S}$ and V respectively. Using the Lie algebra isomorphism $\mathfrak{so}(T_yM) \simeq \wedge^2 T_yM$ and the anti-symmetrization map $q : \wedge^* T_yM \xrightarrow{\sim} \mathbb{C}\mathrm{l}(T_yM)$, we can write

$$\omega_i = \frac{1}{2}q\Gamma_i = \frac{1}{2}q\sum_{i<j}\Gamma_{ij}^k \partial_j \wedge \partial_k$$

where Γ_{ij}^k are the Christoffel symbols of the riemannian connection on TM.

We need to conjugate D^2 by R_λ to get D_λ^2. Recall the Weitzenböck formula [5, Prop.3.18, p.48]:

$$D^2 = \nabla^*\nabla + \frac{\kappa}{4} + q\Omega^F,$$

where κ is the scalar curvature of M, and $q\Omega^F$ is a section of $\mathrm{End}(F) \otimes \mathbb{C}\mathrm{l}(TM)$ obtained by applying the map q to the $\wedge^2 T^*M$ part of the curvature 2-form of the auxiliary bundle F. Since $\nabla^*\nabla = -\sum_{i,j} g^{ij}(\nabla_i\nabla_j - \Gamma_{ij}^k\nabla_k)$, we have

$$D^2 = -\sum_{i,j} g^{ij}(\partial_i + \frac{1}{2}q\Gamma_i + A_i)(\partial_j + \frac{1}{2}q\Gamma_j + A_j) + \sum_{i,j,k} g^{ij}\Gamma_{ij}^k(\partial_k + \frac{1}{2}q\Gamma_k + A_k) + \frac{\kappa}{4} + q\Omega^F.$$

We need to conjugate this by R_λ; this conjugation is equivalent to the rescaling $X \mapsto \lambda X$ combined with the application of U_λ to Clifford algebra elements. Thus,

$$D_\lambda^2 = -\sum_{i,j} g^{ij}(\lambda X)(\partial_i + \frac{1}{2}\lambda U_\lambda q\Gamma_i(\lambda X) + \lambda A_i)(\partial_j + \frac{1}{2}\lambda U_\lambda q\Gamma_j(\lambda X) + \lambda A_j) + \lambda \sum_{i,j,k} g^{ij}(\lambda X)\Gamma_{ij}^k(\partial_k + \frac{1}{2}\lambda U_\lambda q\Gamma_k(\lambda X) + \lambda A_k) + \lambda^2\frac{\kappa}{4} + \lambda^2 U_\lambda q\Omega^F(\lambda X). \tag{2.8}$$

We have written down the dependence on the position X explicitly.

To calculate the limit under $\lambda \to 0+$, the following Taylor series come in handy:

$$g_{ij}(X) = \delta_{ij} + O(|X|^2),$$
$$\Gamma_i(X) = -\frac{1}{4}\sum_{j,a,b} R_{ijab}X^j(\partial_a \wedge \partial_b) + O(|X|^2),$$

where R_{ijab} are the coefficients of the Riemann curvature tensor. Let us write $\Omega_{ij}^M := \frac{1}{2}\sum_{a,b} R_{ijab}\partial_a \wedge \partial_b$. Then

$$\lim_{\lambda\to 0+} D_\lambda^2 = \lim_{\lambda\to 0+}\left(-\sum_{i,j}\delta^{ij}\left(\partial_i - \frac{1}{4}\lambda^2 U_\lambda\big(q\Omega_{ik} + O(|\lambda X|^2)\big)X^k\right)\left(\partial_j - \frac{1}{4}\lambda^2 U_\lambda\big(q\Omega_{jk} + O(|\lambda X|^2)X^k\right) + \lambda^2 U_\lambda q\Omega^F(\lambda X)\right).$$

Now, $q\Omega_{ij}^M$, when expanded in terms of the basis e_I for the Clifford algebra, contains only the terms with $|I| = 2$. Thus, owing to Eq. (1.21), we have

$$\lim_{\lambda\to 0+} \lambda U_\lambda q\Gamma_i(\lambda X) = \lim_{\lambda\to 0+} -\frac{1}{2}\lambda^2 U_\lambda q\Omega_{ij}^M X^j = -\frac{1}{2}\Omega_{ij}^M X^j.$$

Therefore,

$$D_0^2 := \lim_{\lambda\to 0+} D_\lambda^2 = -\sum_i\left(\partial_i - \frac{1}{4}\sum_k \Omega_{ik}X^k\right)^2 + \Omega^F(0). \tag{2.9}$$

Note that $\Omega^F(0) \in \mathrm{End}(S)$. □

2.10 PROOF OF (C). Let k_t^0 be the heat kernel for D_0^2. Coefficients in the asymptotic expansion 1.24 depend continuously on the coefficients of D_λ^2 [1, Thm.2.48, p.98]. Thus, the asymptotic expansion for k_t^0 can be obtained by taking the limit $\lambda \longrightarrow 0+$ of the asymptotic expansion for k_t^λ, which is the expansion 1.24. Owing to Eq. (1.15), we have

$$\operatorname{tr}_s k_t^\lambda(X) \sim \frac{(-2i)^{n/2}}{(4\pi t)^{n/2}} \sum_{j=0}^{\infty} \lambda^{2j-n} \operatorname{tr}_F a(\lambda X)_{j,\{1,\ldots,n\}} t^j. \tag{2.11}$$

We wish to take the limit $\lambda \longrightarrow 0+$, but we are concerned about the coefficients $a(\lambda X)_{j,\{1,\ldots,n\}}$ with $j < n/2$. But they must be zero since we know that taking the limit $\lambda \longrightarrow 0+$ must yield the asymptotic expansion for k_t^0. (In fact, more can be said as we shall see in Cor. 2.14.) Thus,

$$\begin{aligned}\operatorname{tr}_s k_t^0(X) &\sim \lim_{\lambda\to 0+} \frac{(-2i)^{n/2}}{(4\pi t)^{n/2}} \sum_{j \geqslant n/2} \lambda^{2j-n} \operatorname{tr}_F a(\lambda X)_{j,\{1,\ldots,n\}} t^j \\ &= \frac{(-2i)^{n/2}}{(4\pi)^{n/2}} \operatorname{tr}_F a(0)_{\frac{n}{2},(0,1,\ldots,n)}. \end{aligned} \tag{2.12}$$

□

2.13 PROOF OF (D). Write the operator 2.9 as

$$D_0^2 = H + \Omega^F(0).$$

The operators H and $\Omega^F(0)$ commute with each other. So $e^{-tD_0^2} = e^{-tH}e^{-t\Omega^F}$, and the heat kernel of D_0^2 is $k_t^0 = h_t e^{-t\Omega^F}$ where h_t is the heat kernel of H. The operator H is what is called the "generalized harmonic oscillator". Its heat kernel h_t is known [5, Prop.12.25]:

$$h_t(X) = \frac{1}{(4\pi t)^{n/2}} \det{}^{\frac{1}{2}}\left(\frac{t\Omega^M/2}{\sinh t\Omega^M/2}\right) e^{-\frac{1}{4t} g\left(\frac{t\Omega^M}{2}\coth\frac{t\Omega^M}{2}X, X\right)}.$$

□

Remark. Note that the final quantity in Eq. (2.12) is the integrand in Eq. (1.17). So we have obtained a workaround in calculating the integrand for $\operatorname{ind}_s D$, using the rescaled heat kernel k_t^λ instead of the original heat kernel k_t. The key relationship in this vein is that

$$\lim_{t\to 0+} \operatorname{tr}_s k_t(0) = \lim_{\lambda\to 0+} \operatorname{tr}_s k_t^\lambda(X).$$

This follows from applying Cor. 2.14 below to the asymptotic expansion 1.14 of the original heat kernel k_t.

2.14 COROLLARY. *Let $a(X)_{j,I}$ be the coefficients in the asymptotic expansion 1.13. Then*

$$a(0)_{j,I} = 0 \quad \textit{if} \quad j < \frac{|I|}{2} \tag{2.15}$$

Proof. By Eq. (2.3),

$$k_t^0(0) = h_t(0)e^{-t\Omega^F} = \frac{1}{(4\pi t)^{n/2}} \det^{\frac{1}{2}}\Big(\frac{t\Omega^M/2}{\sinh t\Omega^M/2}\Big)e^{-t\Omega^F}. \tag{2.16}$$

Note that this is an element of $(\wedge T_y M) \otimes \mathrm{End}(E)$ by Eq. (1.19). Taking the power series expansion with respect to t,

$$k_t^0(0) = \frac{1}{(4\pi t)^{n/2}} \sum_{j=0}^{\infty} P_j(\tfrac{1}{2}\Omega^M, -\Omega^F)t^j \tag{2.17}$$

where P_j is a homogeneous polynomial of degree j. The above is the asymptotic expansion for $k_t^0(0)$. It has to be equal to, under the limit of $\lambda \longrightarrow 0+$, the asymptotic expansion for k_t^λ. By the expansion 1.24,

$$k_t^\lambda(0) \sim \frac{1}{(4\pi t)^{n/2}} \sum_{j,I} \lambda^{2j-|I|} a(0)_{j,I} e_I t^j. \tag{2.18}$$

Therefore,

$$P_j(\tfrac{1}{2}\Omega^M, -\Omega^F) = \lim_{\lambda\to 0+} \sum_I \lambda^{2j-|I|} a(0)_{j,I} e_I. \tag{2.19}$$

Since the left-hand side is well-defined, the limit in the right-hand side must converge. Hence, $a(0)_{j,I} = 0$ for $j < \frac{|I|}{2}$. □

Remark. In retrospect, Eq. (2.19) can now be written as

$$P_j(\tfrac{1}{2}\Omega^M, -\Omega^F) = \sum_{|I|=2j} a(0)_{j,I} e_I. \tag{2.20}$$

Since $|I| \leqslant n$, we have

$$P_j \neq 0 \quad \text{only if} \quad j = 0, 1, \ldots, n/2.$$

So Eq. (2.17) can be rewritten as

$$k_t^0(0) = \frac{1}{(4\pi t)^{n/2}} \sum_{j=0}^{n/2} P_j t^j. \tag{2.21}$$

2.22 PROOF OF (E). Taking the super-trace on both sides of Eq. (2.21), we get

$$\mathrm{tr}_s\, k_t^0(0) = \frac{1}{(4\pi)^{n/2}} \mathrm{tr}_s\, P_{n/2}(\tfrac{1}{2}\Omega^M, -\Omega^F) = \frac{(-2i)^{n/2}}{(4\pi)^{n/2}} \mathrm{tr}_F\, a(0)_{\frac{n}{2},(0,1,\ldots,n)}, \tag{2.23}$$

where we have used Eq. (2.20) and Eq. (1.15). Comparing Eqs. (2.16) and (2.17), we see that $P_{n/2}(\frac{\Omega^M}{2}, -\Omega^F)$ is the n-form part of

$$\det^{\frac{1}{2}}\left(\frac{\Omega^M/2}{\sinh \Omega^M/2}\right)e^{-q\Omega^F}.$$

We want the super-trace of this. Let $\alpha_1, \ldots, \alpha_n$ be the local coframe for the cotangent bundle dual to the orthonormal frame $e_1, \ldots, e_n$ for TM. Note that

- the power series of $\det^{\frac{1}{2}}\left(\frac{\Omega^M/2}{\sinh \Omega^M/2}\right)$ consists of terms such as $p\alpha_I$ where p is a smooth function;
- the power series of $e^{-q\Omega^F}$ consists of terms such as $A\alpha_{I'}$ where A is a matrix valued smooth function.

When the two terms $p\alpha_I$, $A\alpha_{I'}$ are multiplied, the super-trace of the product is locally of the form

$$\operatorname{tr}_s(pA\alpha_I \wedge \alpha_{I'}) = \operatorname{tr}_F(pA)\operatorname{tr}_{\mathbb{S}}(\alpha_I \wedge \alpha_{I'}) = p\operatorname{tr}_F(A)\operatorname{tr}_{\mathbb{S}}(\alpha_I \wedge \alpha_{I'}).$$

We are only concerned when $|I| + |I'| = n$, in which case the above quantity is equal to

$$(-2i)^{n/2}p\operatorname{tr}_F(A).$$

Multiplying the volume form $\mathrm{vol} = \alpha_1 \wedge \cdots \wedge \alpha_n$ to this gives us $(-2i)^{n/2}(p\alpha_I)(\operatorname{tr}_F(A)\alpha_{I'})$. Our conclusion is that

$$\operatorname{tr}_s P_{n/2}(\tfrac{1}{2}\Omega^M, -\Omega^F)\mathrm{vol} = (-2i)^{n/2}\det^{\frac{1}{2}}\left(\frac{\Omega^M/2}{\sinh \Omega^M/2}\right)\operatorname{tr}_F e^{-\Omega^F}\Big|_{n\text{-form}} \tag{2.24}$$

This is just the product of the characteristic classes

$$\hat{A}(M) = \det^{\frac{1}{2}}\left(\frac{\Omega^M/4\pi i}{\sinh \Omega^M/4\pi i}\right)$$

and

$$\mathrm{ch}(F) = \operatorname{tr}_F(e^{-\Omega^F/2\pi i}),$$

up to a scalar factor. Indeed, making the substitution $\Omega^M \mapsto \Omega^M/2\pi i$ and $\Omega^F \mapsto \Omega^F/2\pi i$ in Eq. (2.24), we get

$$\operatorname{tr}_s P_{n/2}(\tfrac{1}{4\pi i}\Omega^M, -\tfrac{1}{2\pi i}\Omega^F) = (-2i)^{n/2}\hat{A}(M) \wedge \mathrm{ch}(F)\Big|_{n\text{-form}}.$$

Because $P_{n/2}$ is a homogeneous polynomial of degree j,

$$P_{n/2}(\tfrac{1}{2}\Omega^M, -\Omega^F) = (2\pi i)^{n/2}P_{n/2}(\tfrac{1}{4\pi i}\Omega^M, -\tfrac{1}{2\pi i}\Omega^F).$$

Therefore,

$$\operatorname{tr}_s P_{n/2}(\tfrac{1}{2}\Omega^M, -\Omega^F)\mathrm{vol} = (-2i)^{n/2}(2\pi i)^{n/2}\hat{A}(M) \wedge \mathrm{ch}(-\Omega^F).$$

This result, together with Eq. (2.23), proves Eq. (2.4). □

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